Figure 6-12  A neutron of external kinetic energy $K$ incident upon a decreasing potential step of depth $V_0$, which approximates the potential it feels upon entering a nucleus. Its total energy, measured from the bottom of the step potential, is $E$.

drop quite as rapidly at the nuclear surface, in comparison to the de Broglie wavelength, as a step potential.

6-5 THE BARRIER POTENTIAL

In this section we consider a barrier potential, illustrated in Figure 6-13. The potential can be written as follows

$$V(x) = \begin{cases} V_0 & 0 < x < a \\ 0 & x < 0 \text{ or } x > a \end{cases} \quad (6-45)$$

According to classical mechanics, a particle of total energy $E$ in the region $x < 0$, which is incident upon the barrier in the direction of increasing $x$, will have probability one of being reflected if $E < V_0$, and probability one of being transmitted into the region $x > a$ if $E > V_0$.

Neither of these statements describes accurately the quantum mechanical results. If $E$ is not much larger than $V_0$, the theory predicts that there will be some reflection, except for certain values of $E$. If $E$ is not much smaller than $V_0$, quantum mechanics predicts that there is a certain probability that the particle will be transmitted through the barrier into the region $x > a$.

In “tunneling” through a barrier whose height exceeds its total energy, a material particle is behaving purely like a wave. But in the region beyond the barrier it can be detected as a localized particle, without introducing a significant uncertainty in the knowledge of its energy. Thus penetration of a classically excluded region of limited width by a particle can be observed, in the sense that the particle can be observed to be a particle, of total energy less than the potential energy in the excluded region, both before and after it penetrates the region. We shall discuss some consequences of this fascinating effect in the present section, as well as some consequences of the reflection of particles attempting to pass over a barrier. The following section is devoted completely to examples of tunneling through barriers, and considers three of particular importance: (1) the emission of $x$ particles from radioactive nuclei through the potential barrier they experience in the vicinity of the nuclei, (2) the inversion of the ammonia molecule which provides a frequency standard for atomic clocks, and (3) the tunnel diode used as a switching unit in fast electronic circuits.

Figure 6-13  A barrier potential.
For the barrier potential of (6-45), we know from the qualitative arguments of the last chapter that acceptable solutions to the time-independent Schroedinger equation should exist for all values of the total energy \( E \geq 0 \). We also know that the equation breaks up into three separate equations for the three regions: \( x < 0 \) (left of the barrier), \( 0 < x < a \) (within the barrier), and \( x > a \) (right of the barrier). In the regions to the left and to the right of the barrier the equations are those for a free particle of total energy \( E \). Their general solutions are

\[
\psi(x) = Ae^{ik_{1}x} + Be^{-ik_{1}x} \quad x < 0
\]

\[
\psi(x) = Ce^{ik_{2}x} + De^{-ik_{2}x} \quad x > a
\]

where

\[
k_{1} = \frac{\sqrt{2mE}}{\hbar}
\]

In the region within the barrier, the form of the equation, and of its general solution, depends on whether \( E < V_{0} \) or \( E > V_{0} \). Both of these cases have been treated in the previous sections. In the first case, \( E < V_{0} \), the general solution is

\[
\psi(x) = Fe^{-k_{1}x} + Ge^{k_{1}x} \quad 0 < x < a
\]

where

\[
k_{1} = \frac{\sqrt{2m(V_{0} - E)}}{\hbar} \quad E < V_{0}
\]

In the second case, \( E > V_{0} \), it is

\[
\psi(x) = Fe^{k_{2}x} + Ge^{-k_{2}x} \quad 0 < x < a
\]

where

\[
k_{2} = \frac{\sqrt{2m(E - V_{0})}}{\hbar} \quad E > V_{0}
\]

Note that (6-47) involves real exponentials, whereas (6-46) and (6-48) involve complex exponentials.

Since we are considering the case of a particle incident on the barrier from the left, in the region to the right of the barrier there can be only a transmitted wave as there is nothing in that region to produce a reflection. Thus we can set

\[
D = 0
\]

In the present situation, however, we cannot set \( G = 0 \) in (6-47) since the value of \( x \) is limited in the barrier region, \( 0 < x < a \), so \( \psi(x) \) for \( E < V_{0} \) cannot become infinitely large even if the increasing exponential is present. Nor can we set \( G = 0 \) in (6-48) since \( \psi(x) \) for \( E > V_{0} \) will have a reflected component in the barrier region that arises from the potential discontinuity at \( x = a \).

We consider first the case in which the energy of the particle is less than the height of the barrier, i.e., the case:

\( E < V_{0} \)

In matching \( \psi(x) \) and \( d\psi(x)/dx \) at the points \( x = 0 \) and \( x = a \), four equations in the arbitrary constants \( A, B, C, F, \) and \( G \) will be obtained. These equations can be used to evaluate \( B, C, F, \) and \( G \) in terms of \( A \). The value of \( A \) determines the amplitude of the eigenfunction, and it can be left arbitrary. The form of the probability density corresponding to the eigenfunction obtained is indicated in Figure 6-14 for a typical situation. In the region \( x > a \) the wave function is a pure traveling wave and so the probability density is constant, as for \( x > 0 \) in Figure 6-10. In the region \( x < a \) the wave function is principally a standing wave but has a small traveling wave component because the reflected traveling wave has an amplitude less than that of the
incident wave. So the probability density in that region oscillates but has minimum values somewhat greater than zero, as for \( x < 0 \) in Figure 6-10. In the region \( 0 < x < a \) the wave function has components of both types, but it is principally a standing wave of exponentially decreasing amplitude, and this behavior can be seen in the behavior of the probability density in the region.

The most interesting result of the calculation is the ratio \( T \), of the probability flux transmitted through the barrier into the region \( x > a \), to the probability flux incident upon the barrier. This transmission coefficient is found to be

\[
T = \frac{v_1 C*C}{v_1 A*A} = \left[ 1 + \frac{(e^{ik_0a} - e^{-ik_0a})^2}{16 \frac{E}{V_0} \left( 1 - \frac{E}{V_0} \right)} \right]^{-1} = \left[ 1 + \frac{\sinh^2 k_\text{eff} a}{\frac{E}{V_0} \left( 1 - \frac{E}{V_0} \right)} \right]^{-1}
\]  

(6-49)

where

\[
k_\text{eff} a = \sqrt{\frac{2mV_0a^2}{\hbar^2}} \left( 1 - \frac{E}{V_0} \right) \quad E < V_0
\]

If the exponents are very large, this formula reduces to

\[
T \simeq 16 \frac{E}{V_0} \left( 1 - \frac{E}{V_0} \right) e^{-2k_\text{eff}a} \quad k_\text{eff}a \gg 1
\]  

(6-50)

as can be verified with ease. When (6-50) is a good approximation, \( T \) is extremely small.

These equations make a prediction which is, from the point of view of classical mechanics, very remarkable. They say that a particle of mass \( m \) and total energy \( E \), incident on a potential barrier of height \( V_0 > E \) and finite thickness \( a \), actually has a certain probability \( T \) of penetrating the barrier and appearing on the other side. This phenomenon is called barrier penetration, and the particle is said to tunnel through the barrier. Of course, \( T \) is vanishingly small in the classical limit because in that limit the quantity \( 2mV_0a^2/\hbar^2 \), which is a measure of the opacity of the barrier, is extremely large.

We shall discuss barrier penetration in detail shortly, but let us first finish describing the calculations by considering the case in which the energy of the particle is greater than the height of the barrier, i.e., the case:

\( E > V_0 \)

In this case the eigenfunction is oscillatory in all three regions, but of longer wavelength in the barrier region, \( 0 < x < a \). Evaluation of the constants \( B, C, F, \) and \( G \) by application of the continuity conditions at \( x = 0 \) and \( x = a \), leads to the following formula for the transmission coefficient

\[
T = \frac{v_1 C*C}{v_1 A*A} = \left[ 1 - \frac{(e^{ik_0a} - e^{-ik_0a})^2}{16 \frac{E}{V_0} \left( \frac{E}{V_0} - 1 \right)} \right]^{-1} = \left[ 1 + \frac{\sin^2 k_\text{eff} a}{\frac{E}{V_0} \left( \frac{E}{V_0} - 1 \right)} \right]^{-1}
\]

(6-51)
where

$$k_{\text{H}}a = \sqrt{\frac{2mV_0a^2}{\hbar^2} \left( \frac{E}{V_0} - 1 \right)}$$

where $E > V_0$

**Example 6-4.** An electron is incident upon a rectangular barrier of height $V_0 = 10$ eV and thickness $a = 1.8 \times 10^{-10}$ m. This rectangular barrier is an idealization of the barrier encountered by an electron that is scattering from a negatively ionized gas atom in the “plasma” of a gas discharge tube. The actual barrier is not rectangular, of course, but it is about the height and thickness quoted. Evaluate the transmission coefficient $T$ and the reflection coefficient $R$, as a function of the total energy $E$ of the electron.

From Example 6-2 we can see that if $E$ is a reasonable fraction of $V_0$ the penetration length $\Delta x$ will be comparable to the barrier thickness $a$. Thus we can expect appreciable transmission through the barrier. To determine exactly how much, we use the numbers given to evaluate the combination of parameters

$$\frac{2mV_0a^2}{\hbar^2} \approx 2 \times 9 \times 10^{-31} \text{ kg} \times 10 \text{ eV} \times 1.6 \times 10^{-19} \text{ joule/eV} \times (1.8)^2 \times 10^{-20} \text{ m}^2 \approx 9 \text{ joule}^2 \text{ sec}^{-2}$$

which enters (6-49). From this we can plot $T$, and also $R = 1 - T$, versus $E/V_0$, in the range $0 \leq E/V_0 \leq 1$. The plot is shown in Figure 6-15. We see that $T$ is very small when $E/V_0 \ll 1$. But, when $E/V_0$ is only somewhat smaller than one, so that $E$ is nearly as large as $V_0$, $T$ is not at all negligible. For instance, when $E$ is half as large as $V_0$ so that $E/V_0 = 0.5$, the transmission coefficient has the appreciable value $T \approx 0.05$. It is apparent that electrons can penetrate this barrier with relative ease.

For $E/V_0 > 1$, we evaluate $T$, and $R = 1 - T$, from (6-51), using the same combination of parameters as before. The results are also shown in Figure 6-15. For $E/V_0 > 1$, the transmission coefficient $T$ is in general somewhat less than one, owing to reflection at the discontinuities in the potential. However, from (6-51) it can be seen that $T = 1$ whenever $k_{\text{H}}a = \pi, 2\pi, 3\pi, \ldots$. This is simply the condition that the length of the barrier region, $a$, is equal to an integral or half-integral number of de Broglie wavelengths $\lambda_{\text{H}} = 2\pi/k_{\text{H}}$ in that region. For this particular barrier, electrons of energy $E \approx 21$ eV, 53 eV, etc., satisfy the condition $k_{\text{H}}a = \pi, 2\pi, \text{ etc.}$, and so pass into the region $x > a$ without any reflection. The effect is a result of destructive interference between reflections at $x = 0$ and $x = a$. It is closely related to the Ramsauer effect observed in the scattering of low-energy electrons by noble gas atoms, in which electrons of certain energies in the range of a few electron volts pass through these atoms as if they were not there, and so have transmission coefficients equal to one. Essentially the same effect is seen in scattering of neutrons, with energies of a few MeV, from all nuclei. The nuclear effect, called size resonance, will be discussed later in the book.

![Figure 6-15](image)

**Figure 6-15** The reflection and transmission coefficients $R$ and $T$ for a particle incident upon a potential barrier of height $V_0$ and thickness $a$, such that $2mV_0a^2/\hbar^2 = 9$. The abscissa $E/V_0$ is the ratio of the total energy of the particle to the height of the potential barrier.
We can bring together the results of the last three sections by comparing the plot of the energy dependence of the reflection coefficient $R$ for a barrier potential, in Figure 6-15, with the plot of the same thing for a step potential, in Figure 6-11. The comparison shows that for both potentials $R \to 1$ as $E/V_0 \to 0$, and $R \to 0$ as $E/V_0 \to \infty$, with the decrease in $R$ occurring around $E/V_0 = 1$. But for the barrier potential the reflection coefficient approaches one gradually, at small energies, since the finite thickness of the classically excluded region allows some transmission. Also, the barrier potential reflection coefficient oscillates, at large energies, because of interferences in the reflections from its two discontinuities. As the step potential can be considered to be a limiting case of a barrier of very great width, we can see from our comparison the behavior of the barrier potential reflection coefficient in this limit.

Now we shall discuss in some detail the origins of these results. They all involve phenomena which arise from the wavelike behavior of the motion of microscopic particles, and each phenomenon is also observed in other types of wave motion. As we remarked in Chapter 5, the time-independent differential equation governing classical wave motion is of the same form as the time-independent Schroedinger equation. For instance, electromagnetic radiation of frequency $\nu$ propagating through a medium with index of refraction $\mu$ obeys the equation

$$\frac{d^2\psi(x)}{dx^2} + \left(\frac{2\pi\nu}{c}\mu\right)^2 \psi(x) = 0 \quad (6-52)$$

where the function $\psi(x)$ specifies the magnitude of the electric or magnetic field. When we compare this with the time-independent Schroedinger equation, written in the form

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)]\psi(x) = 0$$

we see that they are identical if the index of refraction in the former is connected with the potential energy function in the latter by the relation

$$\mu(x) = \frac{c}{2\pi\nu} \sqrt{\frac{2m}{\hbar^2} [E - V(x)]} \quad (6-53)$$

Thus the behavior of an optical system with index of refraction $\mu(x)$ should be identical to the behavior of a mechanical system with potential energy $V(x)$, providing the two functions are related as in (6-53). Indeed, there are optical phenomena which are exactly analogous to each of the quantum mechanical phenomena that arise in considering the motion of an unbound particle. An optical phenomenon, completely analogous to the total transmission of particles over barriers of length equal to an integral or half-integral number of wavelengths, is used in the coating of lenses to obtain very high light transmissions and in thin film optical filters.

An optical analogue to the penetration of barriers by particles is found in the imaginary indices of refraction that arise in total internal reflection. Consider a ray of light incident upon a glass-to-air interface at an angle greater than the critical angle $\theta_c$. The resulting behavior of the light ray is called total internal reflection, and it is illustrated in the top of Figure 6-16. A detailed treatment of the process in terms of electromagnetic theory shows that the index of refraction, measured along the line $ABC$, is real in the region $AB$ but imaginary in the region $BC$. Note that an imaginary $\mu(x)$ is suggested by (6-53) for a region analogous to one in which $E < V(x)$. Furthermore, electromagnetic theory shows that there are electromagnetic vibrations in the region $BC$ of exactly the same form as the decreasing exponential standing wave of (6-29) for the region where $E < V(x)$. The flux of energy (the Poynting vector) is zero in this electromagnetic standing wave, just as the flux of probability is zero in the quantum mechanical standing wave, so the light ray is totally reflected. However, if a second block of glass is placed near enough to the first block to be in the region in
Figure 6-16  Top: Illustrating total internal reflection of a light ray. The angle of incidence is greater than the critical angle. Bottom: Illustrating frustrated total internal reflection. Some of the light ray is transmitted through the air gap if the gap is sufficiently narrow.

Figure 6-17  The total internal reflection of water waves. A long vibrating plunger on the left produces a set of waves in a region of shallow water, the waves being illuminated so as to make their crests easily visible. The waves are totally internally reflected at the diagonal boundary of a region where the layer of water abruptly becomes deeper, this reflection occurring because the velocity of water waves depends on the depth of the water. Note that the intensity of the waves decreases rapidly when they try to penetrate into the region of deeper water, but there is some penetration of that region. (Courtesy Film Studio, Education Development Center)
which the electromagnetic vibrations are still appreciable, these vibrations are picked up and propagate through the second block. Furthermore, the electromagnetic vibrations in the air gap now carry a flux of energy through to the second block. This phenomenon, called frustrated total internal reflection, is illustrated in the bottom of Figure 6-16. Essentially the same thing happens in the quantum mechanical case when the region in which $E < V(x)$ is reduced from infinite thickness (step potential) to finite thickness (barrier potential). The transmission of light through an air gap, at an angle of incidence greater than the critical angle, was first observed by Newton around 1700. The equation relating the intensity of the transmitted beam to the thickness of the air gap, and other parameters, is identical in form to (6-49), and it has been verified experimentally.

It is particularly easy to observe frustrated total internal reflection of electromagnetic waves, using the microwave region of the spectrum and two blocks of paraffin separated by an air gap. Furthermore, careful inspection of the "ripple tank" photographs in Figures 6-17 and 6-18 will show that the phenomenon can even be observed with water waves. Frustrated total internal reflection, or its quantum mechanical equivalent barrier penetration, arises from properties common to all forms of classical or quantum mechanical wave motion.

6-6 EXAMPLES OF BARRIER PENETRATION BY PARTICLES

There are a number of interesting, and important, examples of barrier penetration by microscopic particles. A widespread, but not widely recognized, example occurs in aluminum household wiring. The usual way for an electrician to join two wires is to twist them together. Often there is a layer of aluminum oxide between the two wires, and this material is quite an effective