Inventory systems for independent demand

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Inventory systems for independent demand

- The models for the management of inventory systems for independent demand
  - The EOQ-ROP model
  - The IE model
  - The safety stocks model

- With reference to a specific warehouse (or collection point), stock management models determine:
  - What to order
  - How much to order
  - When to order

Re-orders of product i during period t \((R_{it})\) are decided only knowing the level of \(S_{it}\) inventory, independently from demand \((D_{it})\).
Inventory systems for independent demand

- Basic assumption of the models for inventory management

  - Constant consumption over time \( D_{it} \) [unit/year]
  - Constant ordering cost \( a \) [€]
  - Constant value of the good \( p \) [€/pieces]
  - Constant ownership rate \( c_m \) [€ / € x year]
  - Constant provisioning lead time \( LT \) [days]
  - Pre-defined working calendar \( H \) [days / year]
  - Pre-defined productive rhythm \( r \) [pieces / day]
  - Infinite capacity of warehouse
  - Constant freight cost

short movie
Inventory systems for independent demand

- Agenda:
  the models for the management of inventory systems for independent demand

  - The EOQ-ROP model
    - Economic Order Quantity – Re-Order Point (Model)

  - The IE model
    - Economic Interval (Model)

  - The safety stocks model
The EOQ-ROP model

- **Objectives**
  1. Identifying the quantity $q$ [pieces] to re-order that minimizes the total cost, sum of the:
     - ordering cost,
     - purchasing cost
     - and (stock) holding cost,
  2. and the conditions that determine the orders issuing

- **Characteristics**
  - Variable interval of orders issuing
  - Fixed quantity ordered
  - Continuous control
  - Independent entries reorder
The EOQ-ROP model

- The saw-tooth diagram

   - Fixed quantity Q is always re-ordered
   - The downstream consumption is constant over time

   - When inventory level falls below the re-order point, a new lot is ordered
   - The new lot arrives after LT days

   Inventory level
   
   Q
   Q/2
   ROP
   LT
   time
The EOQ-ROP model

- The cost function

  - Cost of purchasing or production
    \[ p \times D \]

  - Cost of holding stocks
    \[ p \times c_m \frac{x}{q} / 2 \]

  - Cost of order issuing
    \[ a \times \frac{D}{q} \]

- Overall cost function

  \[ F = p \times D + p \times c_m \frac{x}{q} / 2 + a \times \frac{D}{q} \]
The EOQ-ROP model

- A note on $c_m$
  - $c_m$ parameter represents the time value of money and it is differently named according to the different environments in which it is used
    - Ownership rate
      - When talking about stocks and inventories
    - Interest rate
      - When talking about lending or borrowing money at financial institutions (or markets)
    - Discount rate
      - When calculating (net) present values (NPVs)
    - Rate of return
      - When calculating future values and within the investment appraisal area
    - Opportunity cost of capital
      - When talking about capital budgeting and in the investment appraisal area
  
  - From a mere financial viewpoint, investing 1 Euro in stocks is **never** profitable (i.e. it yields a negative NPV)
    - For this reason stocks are (or should be) considered as service-related management levers

\[
NPV = -1\€ + \frac{1\€}{(1+i)^n} < 0 \quad \forall i, n
\]

Amount of money invested today

Amount of money expected after n periods
The EOQ-ROP model

- The cost function
  - The Economic Order Quantity (EOQ) is so that $\frac{\partial F}{\partial q} = 0$

\[ p \times D + \frac{p \times c_m \times q}{2} + \frac{a \times D}{q} = 0 \]

\[ \text{EOQ} = \sqrt{\frac{2 \cdot a \cdot D}{p \cdot c_m}} \]

- The re-order point (ROP)

\[ \text{ROP} = LT \times \frac{D}{H} \]

D/H represents daily demand or demand rate.
Model improvements: relaxing 3 basic hypotheses

- Hypothesis 1: constant consumption over time and constant provisioning lead time
  - It leads to consider safety stocks (see later for details)

![Graph showing inventory levels and safety stocks]

\[ \text{EOQ}^* = \sqrt{\frac{2 \cdot a \cdot D}{p \cdot c_m}} = \text{EOQ} \]

\[ \text{ROP}^* = \frac{\text{LT} \times D}{H} + \text{SS} = \text{ROP} + \text{SS} \]
The EOQ-ROP model

- Model improvements: relaxing 3 basic hypotheses
  - Hypothesis 2: infinite productive rhythm
    - EOQ under finite production rate ($r$) can be calculated by setting reference to the “geometry” of saw-tooth diagram

\[
\text{maximum inventory} = T \cdot \tan \alpha - T \cdot \tan \beta = T \cdot \tan \alpha \cdot \left(1 - \frac{\tan \beta}{\tan \alpha}\right) = q \cdot \left(1 - \frac{D}{H \cdot r}\right)
\]

Production rate $= r = \tan \alpha$
Consumption rate $= D/H = \tan \beta$

\[
EOQ^* = \sqrt{\frac{2 \cdot a \cdot D}{p \cdot c_m} \cdot \left(1 - \frac{D}{H \cdot r}\right)} = \sqrt{\frac{1}{1 - \frac{D}{H \cdot r}}} = EOQ \cdot \mu
\]
The EOQ-ROP model

- Model improvements: relaxing 3 basic hypotheses
  - Hypothesis 3: constant cost of purchasing / production
    - It allows to consider quantity discounts
    - It leads to consider a more complex cost function to derive, since “p” parameter changes (i.e. decreases) when q increases:

\[
p = \begin{cases} 
  p_1 & \text{for } q < q_1 \\
  p_2 < p_1 & \text{for } q_1 < q < q_2 \\
  p_3 < p_2 & \text{for } q > q_2 
\end{cases}
\]
Inventory systems for independent demand

- **Agenda:**
  - the models for the management of inventory systems for independent demand
    - The EOQ-ROP model
    - The IE model
    - The safety stocks model
The IE model

- **Objective**
  - Identifying (at the end of each IE time period) the quantity to re-order that allows the **availability** of each product to achieve again a pre-defined level, called objective level (OL)

```
Availability = Physical inventory level + Orders in progress - Reserved stocks - Safety stocks
```

- **Characteristics**
  - Fixed interval (IE) of orders issues
  - Variable ordered quantity
  - Discontinuous control
  - Independent or combined entries reorder

How many pieces you can see and touch in the warehouse

How many pieces have just been ordered upstream, but have not reached the warehouse yet

How many pieces have just been promised to a customer downstream, but that are still in the warehouse

Pieces “untouchable”, stored to protect the system against unpredictable events
The IE model

- The saw-tooth diagram

Availability is modified by the issuing of orders at the end of each IE period

The physical inventory level is modified accordingly by the arrival of pieces after LT
The IE model

- The objective level
  - It must cover a time fence as long as IE + LT periods

\[ \text{OL} = (\text{IE} + \text{LT}) \times \frac{D}{H} \]

Overall period to cover

\[ \text{OL}^* = (\text{IE} + \text{LT}) \times \frac{D}{H} + \text{SS} \]

Horizon covered by OL

Horizon covered by OL

Inventory level
The IE model

- SW(OT) analysis assuming EOQ-ROP as benchmark
  - IE model:
    - Easily allows joint (combined entries) re-orders
    - Allows for discontinuous control, which leads to lower cost of control
    - Unfortunately it does not optimize any (cost) function

- IE is typically selected to manage products with low stock-holding and re-order costs
- EOQ-ROP is adopted for products with high purchasing / production costs, preferably when they are not synchronized with other products
Inventory systems for independent demand

- Agenda: the models for the management of inventory systems for independent demand
  - The EOQ-ROP model
  - The IE model
  - The safety stocks model
The SS model

- Basic principles
  - Safety stocks protect the inventory management system against **unpredictably** high downstream consumption (demand) and upstream lead time disruptions
  - Safety stocks represent a “virtual” inventory level, useful for management-oriented purposes
    - Therefore they do not correspond to a physical inventory level
  - Safety stocks are “untouchable” and so they are subtracted from the availability
    - Furthermore – to ensure an adequate service level over time – they have also to be re-stored as soon as they are consumed

- Objective
  - Properly dimensioning safety stocks, by considering both consumption and lead time as random variables
    - i.e. they are provided with their own mean value and variance value

\[ D_i, \sigma_i, LT_i, \sigma_{LTi} \]
The SS model

- The saw-tooth diagram

Safety stocks are useful to prevent from stock-out when an order has been already issued, but the corresponding quantity (pieces) has not been reached the warehouse yet.
The SS model

- The general calculation formula
  - The service level factor ($k$) is calculated by using the **cumulative Normal standard distribution** for random variables ($\Phi$, also called Gauss function, provided by tables) that corresponds to the required service level ($SL$):

$$\Phi(k) = \int_{-\infty}^{k} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = SL$$

- A criticism
  - The service level is measured in terms of stock-out frequency
    - No reference is set to the **stock-out quantity**, which is often more important than frequency, especially in manufacturing (B2B) environments, where the time to recover from stock-out is crucial

<table>
<thead>
<tr>
<th>Period (week)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand (pieces)</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Inventory on hand - A</td>
<td>100</td>
<td>99</td>
<td>99</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Inventory on hand - B</td>
<td>100</td>
<td>10</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

- On the basis of the service level, case B seems to be preferable, while experience suggests that case A is much easier to recover

![Diagram](image)

- $SS = k \cdot \sigma^*_D$

- **Service level factor “Combined” standard deviation of demand during LT**

- $SL = 90\% \Rightarrow k \approx 1.281$
- $SL = 95\% \Rightarrow k \approx 1.645$
- $SL = 99\% \Rightarrow k \approx 2.326$
The SS model

- Calculating the combined standard deviation of demand during lead time
  - Broadly speaking
    \[ \sigma_D^* = \left( \sigma_{D,LT}^2 + \sigma_{LT,D}^2 \right)^\mu \]
  - The most popular case is where demand and lead time are completely not-correlated, i.e. \( \mu = 0.5 \)

\[ \sigma_D^* = \sqrt{\sigma_D^2 \cdot LT + \sigma_{LT}^2 \cdot D^2} \]

- It takes into account the (statistical) correlation between consumption and LT
- Area where the replenishment can take place due to the combined variance of demand and lead time
The SS model

**A criticism**

- The calculation model of SS is variance-based
  - Whenever the (demand) variance is unable to express in a right way the variability (e.g. in the case of *lumpy* demand), the SS calculation model is **useless**
    - Lumpy demands are typical of e.g. promotions at the retail level, C-class products, demand resulting from orders after a tender has been won etc.
  - Even by having to resort to a remarkably high level of safety stocks, the production system cannot be (effectively) protected against the peak
    - In this case safety stocks are at the same time **costly** and **useless**

**Example:**

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Average</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>1000</td>
<td>175</td>
<td>259,8076211</td>
</tr>
<tr>
<td>Steady demand</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>Peak demand</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>900</td>
<td>75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Service level</th>
<th>&quot;k&quot;</th>
<th>Safety factor stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>90%</td>
<td>1.28</td>
<td>333</td>
</tr>
<tr>
<td>95%</td>
<td>1.64</td>
<td>427</td>
</tr>
<tr>
<td>99%</td>
<td>2.33</td>
<td>604</td>
</tr>
<tr>
<td>99.9%</td>
<td>3.09</td>
<td>803</td>
</tr>
<tr>
<td>99.99%</td>
<td>3.72</td>
<td>966</td>
</tr>
</tbody>
</table>

**"Normal" service levels yield safety stocks useless to face the peak (600 vs. 900)**

To "manage" the peak through the (traditional) safety stocks, the required service level is completely unrealistic.

During week 10 a peak takes place. The height of the peak is 900 pieces.

The peak (alone) is responsible for the whole demand variance.
### Executive summary

- Inventory systems for independent demand are models where the re-orders of each product during each planning period are decided only knowing the level of inventory, independently from demand.
- Economic Order Quantity – Re-Order Point (EOQ-ROP) is the most popular model and it is usually adopted for products with high purchasing / production costs, preferably when they are not synchronized with other products.
- Fixed Interval (IE) model is typically selected to manage products with low stock-holding and re-order costs, since it does not optimize any (cost) function. However, it allows joint (combined entries) re-orders and it also allows for discontinuous control, which leads to lower cost of control.
- Safety stocks (SS) are used to protect the inventory management system against unpredictably high downstream consumption (demand) and upstream lead time disruptions.
Inventory systems for independent demand

- Further (suggested) readings
Inventory systems for independent demand

- Practice

1. Company Alpha requires a re-calculation of the Economic Order Quantity (EOQ) and of the Re-Order Point (ROP) for product Beta so as to achieve a 95% service level, given the time series of (independent) demands during the last 10 weeks (see the table, where demand is expressed in thousands).

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>20</td>
<td>30</td>
<td>25</td>
<td>35</td>
<td>30</td>
<td>25</td>
<td>30</td>
<td>20</td>
<td>35</td>
<td>25</td>
</tr>
</tbody>
</table>

In addition, the stock replenishment lead time accounts for 4 weeks, the ownership rate accounts for 12% (per year) and the set-up cost accounts for 300 Euros per set-up.

Finally, the accounting system provides the following data: labor cost equals to 1 Euro per unit (of Beta); raw materials cost equals to 2 Euros per unit; energy cost equals to 1 Euro per unit; depreciation of machinery equals to 1,000,000 Euros per year; overhead costs equal to 250,000 Euros per year.
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Practice

2. Company Alpha operates on a basis of 52 weeks per year (5 days per week) and it considers (independent) demand data of product Gamma (reported in the table), managed through IE model every 10 days.

<table>
<thead>
<tr>
<th>Week</th>
<th>Demand (pieces)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>35</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
</tr>
<tr>
<td>6</td>
<td>45</td>
</tr>
<tr>
<td>7</td>
<td>55</td>
</tr>
<tr>
<td>8</td>
<td>60</td>
</tr>
<tr>
<td>9</td>
<td>65</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

Replenishment lead time accounts for 20 days, while the (yearly) hurdle rate and the service level equal to 15% and 98% respectively.

In addition, the variable production cost accounts for 5 Euros per piece and set-up cost accounts for 10 Euros (per set-up).

You are required to calculate:

- The target (objective) level of inventory
- The safety stocks level
- The average inventory level
Inventory systems for independent demand

- Practice (short discussion):
  1. Average demand (over 10 weeks) equals to 27,500 pieces per week, while the standard deviation of demand equals to 5,124 pieces per week. The overall demand accounts for 275,000 pieces. Parameter $p$ equals to 4 Euros per piece ($1 + 2 + 1$) and $c_m$ parameter equals to 0.0023 (i.e. 0.12/52), since it must be expressed on a weekly basis (compounding is overlooked for the sake of simplicity). As a consequence EOQ accounts for (around) 133,921 pieces. Safety stocks accounts for (around) 17,000 pieces (i.e. $1.65 \times 5,124 \times 2$, namely the square root of 4); as a consequence ROP equals to 127,000.
  2. Consider firstly safety stocks. To this purpose: (i) average demand equals to 50 pieces per week; (ii) the standard deviation of demand equals to 9.5 pieces per week; (iii) the service level factor (corresponding to 98%) accounts for (around) 2.06; (iv) the overall lead time to cover accounts for $10 + 20 = 30$ days (i.e. 6 weeks). So safety stocks equal to $9.5 \times 2.06 \times \sqrt{6} = 48$ pieces. As a consequence the target level equals to 348 pieces (i.e. $50 \times 6 + 48$) and the average inventory level equals to $D \times IE / 2 + SS = 50 \times 2/2 + 48 = 98$ pieces.